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The Principle of Equivalence, Electrodynamics and General Relativity

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Abstract

It is shown that the customary covariant formulation of electrodynamics in General Relativity is incompatible with the Einstein Principle of Equivalence. This is demonstrated for the case of a resistanceless current-carrying wire in a static spherically symmetric gravitational field—where the Einstein Principle of Equivalence implies the existence, in the vicinity of the wire, of a non-zero component of the electric field parallel to the wire, whereas the covariant form of Maxwell's equations does not. An experiment, involving a superconducting current-carrying wire segment placed in the Earth's gravitational field, is suggested. Whether or not a component of electric field parallel to the wire, at a point in the wire's vicinity, would be detected would resolve the issue.

1. *Introduction*

Since its inception the Principle of Equivalence has been given many interpretations. In fact, it has even been categorically denied by some (Synge, 1960). Difficulties with this principle seem particularly acute in conjunction with electromagnetic phenomena since it appears that various phenomena of this type are incompatible with some of the diverse statements of the principle. In order to put the primary considerations of this work into the proper context it will be well here to briefly review several of these difficulties: (a) according to one version† of the equivalence principle a charged particle at rest in a uniform gravitational field should be indistinguishable from one at rest in a system suitably accelerating relative to an inertial frame. However, this seems unsatisfactory since in the former case the electron does not radiate (relative to an observer supported in the field) and in the latter case it does radiate (relative to an inertial system which is momentarily at rest relative to the electron) (Rohrlich, 1963); (b) an electron in free fall in a uniform gravitational field (or even an arbitrary gravitational field) should be indistinguishable from one permanently at rest in an inertial system according to another interpretation of the equivalence principle.‡ Again, however, this seems unsatisfactory since

† This may be called the historical version of the principle. See, e.g., Einstein, A. *et al. The Principle of Relativity*, p. 100. Dover Publications, New York; and for a slightly more modern statement, Sciama, D. W. (1962). *Recent Developments in General Relativity*, p. 432—The MacMillan Co., New York.

‡ This is sometimes called the 'strong' principle of equivalence. See Dicke, R. H. (1964). *Gravitation and Relativity*, p. 13. W. A. Benjamin, Inc., New York.

in the former case the electron supposedly radiates (Fulton & Rohrlich, 1960) (relative to a co-moving inertial observer) while in the latter case it does not.

Some investigators† claim to have resolved these apparent discrepancies with the assertions that one cannot observe radiation locally (even when just fictitious gravitational fields are present) or that two observers (both in flat space) in relative acceleration may not agree on whether or not radiation is present.

For the sake of completeness it should be mentioned here that recently, Dewitt & Brehme (1960) and Rohrlich & Winicour (1966) have derived the equations of motion of a radiating charge in the presence of a permanent gravitational field. Accepting their basic assumptions one must conclude that the electron's motion does explicitly depend on the curvature so that the electron would not (even locally) have the same motion in a permanent gravitational field and an inertial field. This would seem to imply that the principle of equivalence (between such permanent gravitational fields and inertial fields) is definitely not valid—at least for certain electromagnetic phenomena.

In line with these considerations, the present work is concerned with an instance in which a version of the equivalence principle is in contradiction with a prediction of the General Theory. Specifically, it is demonstrated that the customary covariant formulation of electrodynamics in General Relativity is incompatible with the Einstein statement of the Principle of Equivalence, which we take as follows: any experiment involving a region of space of linear size ζ performed in a gravitational field of local (Gaussian) curvature K, must yield the same result when the experiment is performed instead in a system suitably accelerating relative to an inertial frame, in the limit as $\zeta/K \to 0$. That is, the difference between the two results, divided by the value of either, must go to zero in this limit. The incompatibility is demonstrated through the proof that, in the neighborhood of a resistanceless‡ current-carrying wire, the component of the electric field parallel to the wire takes on values of a different order of magnitude depending on whether the wire is in a gravitational field or is considered to be accelerating relative to an inertial system. Furthermore, this result holds in the limit of very weak gravitational fields. This result is felt to be significant since, in the General Theory, the electrodynamics governing the fields outside of current-carrying wires is supposedly on sound footing—perhaps more so than that governing the motion of radiating charges. Further, it is not difficult, as will be discussed, to construct a thought experiment to decide the issue in question.

The format of the ensuing discussion is then as follows: In the next

† See Rohrlich, F. (1963). *Annals of Physics*, **9**, 499; Fulton, T. & Rohrlich, F. (1960); Bondi, M. and Gold, T. (1955). *Proceedings of the Royal Society*, **A299**, 416.

‡ The wire is taken to be resistanceless where, according to Newtonian physics, the ambient electric field would be zero. The electric field under question here is then a result of non-Newtonian considerations.

section we derive the expression for the electric field in the neighborhood of a 'long', 'straight' resistanceless current-carrying wire in a static, spherically symmetric gravitational field utilizing the previously defined Principle of Equivalence. This involves the evaluation of the E field due to an accelerating current-carrying wire—which is done rigorously in the Appendix.

In Section 3, the E field in the neighborhood of a resistanceless current-carrying wire at rest in the above gravitational Schwarszchild field is calculated using the covariant form of Maxwell's equations.

In Section 4, the discrepancy is considered and a thought experiment to resolve the disagreement is suggested.

2. *Equivalence and the Wire*

Consider a resistanceless straight wire of length l carrying a constant current i. The Principle of Equivalence which we have adopted then states that the electric field (in particular the component parallel to the wire) at a distance $d \ll l$ from the wire should be the same when the wire is at rest a sufficiently great distance from a gravitating mass (like the Earth) or when it is suitably accelerating relative to an inertial frame. We shall presently consider the latter case.

In the present section then we are interested in the E field due to an accelerating wire which is carrying a constant current i. The following consists of a somewhat oversimplified treatment of this problem. The exact treatment is considered in the Appendix.

Consider then a straight wire segment of length l carrying a constant current i which has a constant (intrinsic) acceleration g. Let q denote a positive charge which has the same acceleration as the wire and remains a fixed distance $d \ll l$ from the wire. Let S_0 denote an inertial frame relative to which the wire is momentarily at rest, at time t_0 (in S_0). Now the charge will, at each moment, be encountering a field due to 'light pulses' from the wire occurring at earlier times—i.e. encountering the retarded field due to the wire. Now if l and g are not too large (this will be quantified subsequently) we expect that the B field 'emitted' by the wire (relative to S_0) is approximately that given by Ampere's Law for the static situation, namely:

$$B = \frac{2i}{Cd} \tag{2.1}$$

where here, and in all that follows, Gaussian units are used. However, by the time the charge q encounters this field it will have a velocity, v, relative to S_0 given by

$$v \cong g\frac{d}{C} \tag{2.2}$$

Therefore, as seen by S_0, a Lorentz force will act on q, of magnitude

$$F = \frac{1}{C} qvB \cong \frac{2igq}{C^3} \tag{2.3}$$

This force (per unit charge) will then be interpreted as an *electric* field in the rest frame of the charge, given by

$$E \cong \frac{2gi}{C^3} \tag{2.4}$$

with a direction anti-parallel to the direction of the conventional current in the wire.

As stated, this argument is somewhat heuristic. The more exact calculation carried out in the Appendix shows that we should have instead,

$$E = \frac{\pi g i}{C^3} \tag{2.5}$$

under the following conditions:

$$\frac{gl}{C^2} \ll 1; \qquad d \ll l, \qquad \frac{gl}{C^2}\frac{l}{d} \ll 1 \tag{2.6}$$

One sees that these inequalities are well satisfied for the realistic values; $l \sim 10^2$ cm, $d \sim 1$ cm, $g \sim 10^3$ cm/sec^2.

If the Principle of Equivalence were correct then the above result implies that, in the neighborhood of a current-carrying wire in a weak gravitational field (say that of the Earth), one should find an electric field, anti-parallel to the current in direction and in magnitude given by

$$E = \frac{\pi g i}{C^3} = \frac{\pi i}{C^3}\frac{GM}{R^2} \tag{2.7}$$

This is a prediction which, in principle, can be tested—a point which will be discussed later.

We shall find in the next section, however, that when this calculation is carried out with the generally covariant form of Maxwell's equations applied to a current-carrying wire at rest in a gravitational field we do not find an electric field of the same order of magnitude as that predicted by equation (2.7).

The generally covariant Maxwell's equations are thus incompatible with the Principle of Equivalence.

3. *Wire in a Gravitational Field*

In this section we shall calculate the electric field (component parallel to the wire) in the vicinity of a resistanceless current-carrying wire segment placed at a large Schwarszchild distance r from the center of a gravitating mass (the Earth). It will be shown that this field is zero, at the point of interest, to terms of first order in GM/rC^2, thus contradicting the Principle of Equivalence.

In the following calculation we need only consider the contribution to the metric due to the gravitating mass—i.e. we may neglect the contribution coming from the current-carrying wire itself. This may be seen as follows:

From the solution to Einstein's equations for weak fields† we have the following expression for the contribution to the metric $g_{\mu\nu}^{\text{elm}}$ due to the electromagnetic field

$$g_{\mu\nu}^{\text{elm}} = \eta_{\mu\nu} - \frac{4G}{C^4}\int \frac{[T_{\mu\nu}^{\text{elm}}]}{r'} d^3x \tag{3.1}$$

where $\eta_{\mu\nu}$ is the diagonal matrix $(-1,-1,-1,1)$ for Minkowski space; $T_{\mu\nu}^{\text{elm}}$ signifies the electromagnetic stress tensor, and the brackets signify that retarded values are used.

As a typical element we have

$$g_{44}^{\text{elm}} = 1 - \frac{4G}{C^4}\int \frac{T_{44}^{\text{elm}}}{r'} d^3x = 1 - \frac{4G}{C^4}\int \frac{(E^2+H^2)}{8\pi r'} d^3x \tag{3.2}$$

where the retardation brackets have been dropped since we have a stationary situation.

Now to get an estimate of the above integral we replace the wire segment by a circular ring of radius R (proportional to the length of the wire segment) and cross-sectional area a, carrying a current i. For this case‡

$$U \equiv \frac{1}{8\pi}\int (E^2+H^2)\, d^3x = \frac{2\pi R}{C^2}\left\{-\frac{7}{4} + \ln\frac{8R}{a}\right\} i^2 \cong \frac{Ri^2}{C^2} \tag{3.3}$$

We now assume that

$$\int\limits_{\substack{\text{straight}\\ \text{line}}} \frac{(E^2+H^2)}{8\pi} d^3x \cong \int\limits_{\substack{\text{circ.}\\ \text{wire}}} \frac{(E^2+H^2)}{8\pi} d^3x \cong \frac{Ri^2}{C^2} \tag{3.4}$$

For the straight wire segment we then have

$$\int \frac{(E^2+H^2)}{8\pi r'} d^3x \lesssim \frac{1}{d}\int \frac{(E^2+H^2)}{8\pi} d^3x \cong \frac{Ri^2}{dC^2} \tag{3.5}$$

where d is the distance of the field point from the wire.

Therefore,

$$g_{44}^{\text{elm}} \cong 1 - \frac{4GRi^2}{dC^6} \tag{3.6}$$

† See, for instance, Tolman, R. C. (1934). *Relativity, Thermodynamics and Cosmology*, p. 238. Clarendon Press, Oxford.

‡ See, for instance, Abraham, M. and Becker, R. (1930). *The Classical Theory of Electricity and Magnetism*, p. 176. Hafner Publishing Co., Inc., New York.

Again, using the weak field solution, we have for the contribution of the gravitating mass to the metric, the expression

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{4G}{C^4} \int \frac{\{T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\}}{r'} d^3 x \tag{3.7}$$

Therefore,

$$g_{44} \cong 1 - \frac{2G}{C^4} \int \frac{C^2 \rho \, d^3 x}{r'} = 1 - \frac{2GM}{rC^2} \tag{3.8}$$

where in the integral in equation (3.7) the $g_{\mu\nu}$ are given their Minkowski values, and we assume that $T \cong T_{44}$.

Therefore, as can be seen by comparing equations (3.8) and (3.6), a sufficient condition that the electromagnetic contribution to the metric is negligible is that

$$\frac{Ri^2}{MC^4} \ll \frac{d}{r} \tag{3.9}$$

If we take the mass of the Earth for M, and the radius of the Earth for r, then for these realistic values, $r \sim 10^8$ cm, $d \sim 1$ cm, $i \sim 10^9$ statamp, $M \sim 10^{28}$ gm, $R \sim 10^2$ cm, we obtain

$$\frac{Ri^2}{MC^4} \sim 10^{-50}, \qquad \frac{d}{r} \sim 10^{-8}$$

and so the inequality of equation (3.9) is indeed well satisfied and we can safely neglect all electromagnetic contributions to the metric.

We now proceed to the calculation of E near the wire.

In the following, a semicolon indicates covariant differentiation and a comma ordinary differentiation. Two indices following a semicolon or comma indicate second covariant or ordinary derivatives, respectively.

We take the generally covariant form of Maxwell's equations as

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} = A_{\mu;\nu} - A_{\nu;\mu}, \qquad F^{\mu\nu}_{;\nu} = \frac{4\pi}{C} j^\mu \tag{3.10}$$

where A^μ is the 4-vector potential, which shall be selected in the Lorentz gauge $A^\mu_{\mu;} = 0$, and j^μ is the 4-current density.†

Putting the second of equations (3.10) into a form depending on j_μ, and utilizing the first of equations (3.10) yields

$$\frac{4\pi}{C} j_\mu = g^{\alpha\nu}\{A_{\mu;\alpha\nu} - A_{\alpha;\mu\nu}\} \tag{3.11}$$

Using the general relation, $A_{\lambda;\epsilon\beta} - A_{\lambda;\beta\epsilon} = A_\sigma B^\sigma_{\lambda\epsilon\beta}$, where $B^\sigma_{\lambda\epsilon\beta}$ is the Riemann Christoffel tensor, then yields the following relation

$$g^{\alpha\beta} A_{\mu;\beta\alpha} + g^{\alpha\beta} B^\epsilon_{\beta\alpha\mu} A_\epsilon = \frac{4\pi}{C} j_\mu \tag{3.12}$$

† See Tolman, R. C. (1934). *Relativity, Thermodynamics and Cosmology*, p. 258. Clarendon Press, Oxford.

where the Lorentz gauge condition has been used.

The metric involved here depends, due to the preceding discussion, only on that produced by the gravitating mass. Further, since the relation

$$g^{\alpha\beta} B^{\epsilon}_{\beta\alpha\mu} = 0 \tag{3.13}$$

is equivalent to the field equations (due to just gravitating matter) in empty space, we can immediately re-write equation (3.12) as

$$g^{\alpha\beta} A_{\mu;\beta\alpha} = \frac{4\pi}{C} j_\mu \tag{3.14}$$

Now we are considering a region of the gravitational field which is to be very weak, so we put

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \qquad h^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta} \tag{3.15}$$

where the Minkowski metric, $\eta_{\mu\nu}$, is the diagonal matrix $(-1,-1,-1,1)$, and $h_{\mu\nu}$ and derivatives of $g_{\mu\nu}$ (or $h_{\mu\nu}$) are to be regarded as small quantities of the first order. In the following we shall only be interested in relations to first order in these small quantities. We shall presently write out equation (3.14) to first order, which becomes

$$\Box A_\mu - \eta^{\alpha\sigma}\eta^{\beta\epsilon} h_{\sigma\epsilon} A_{\mu,\beta\alpha} - 2\eta^{\alpha\beta}\eta^{\sigma\lambda}[\mu\beta,\lambda]^h A_{\sigma,\alpha}$$

$$- \eta^{\alpha\beta}\eta^{\sigma\lambda}[\mu\beta,\lambda]^h_{,\alpha} A_\sigma - \eta^{\alpha\beta}\eta^{\sigma\lambda}[\beta\alpha,\lambda]^h A_{\mu,\sigma} = \frac{4\pi}{C} j_\mu \tag{3.16}$$

where

$$\Box(\ldots) \equiv \eta^{\alpha\beta}(\ldots)_{,\alpha\beta} \qquad \text{and} \qquad [\mu\alpha,\lambda]^h \equiv \tfrac{1}{2}(h_{\mu\lambda,\alpha} + h_{\alpha\lambda,\mu} - h_{\mu\alpha,\lambda}) \tag{3.17}$$

Now we introduce asymptotic Schwarszchild co-ordinates, so that the invariant arc-length, $d\tau^2$, is given by

$$d\tau^2 = -\left(1 + \frac{2GM}{rC^2}\right)\sum_{i=1}^{3} dx^i\,dx^i + \left(1 - \frac{2GM}{rC^2}\right) dx^4\,dx^4 \tag{3.18}$$

where

$$r^2 = \sum_{i=1}^{3} x^i x^i, \qquad \text{and} \qquad x^4 = Ct \tag{3.19}$$

Therefore, $h_{\alpha\beta}$ is the diagonal matrix consisting of $-2GM/rC^2$ for each diagonal term.

Because of the static nature of the situation it will only be necessary to consider equation (3.14) for $\mu = 4$. Using the corresponding expression from equation (3.16) we then have

$$\left(\frac{2GM}{rC^2} - 1\right)\nabla^2 A_4 - \frac{GM}{r^3 C^2}\mathbf{r}\cdot\nabla A_4 = \frac{4\pi}{C} j_4 \tag{3.20}$$

where we have used the ordinary vector notation:

$$\nabla^2 A_4 \equiv \sum_{i=1}^{3} A_{4,ii} \qquad \text{and} \qquad \mathbf{r}.\nabla A_4 = \sum_{i=1}^{3} x^i A_{4,i} \tag{3.21}$$

and where the static aspect of the situation has been taken into account.

Retaining terms of first order in GM/rC^2 we then have the relation

$$\nabla^2 A_4 + \frac{GM}{r^3 C^2}\mathbf{r}.\nabla A_4 = -\frac{4\pi}{C}\left(1 + \frac{2GM}{rC^2}\right) j_4 \tag{3.22}$$

This is the equation governing the electric field in the neighborhood of the wire.

3a. *Solution of the Wire Equation*

In this section we propose to carry the solution of equation (3.22) through to the extent that it becomes apparent that the component of the electric field parallel to the wire, at a nearby point equidistant from the ends of the wire, is zero.

The physical situation is as follows: The current-carrying wire segment is of co-ordinate length l and is 'straight'—i.e. lies along the x^1-axis, with its center at $x^1 = 0$, and with its center at a co-ordinate distance r_0 from the center of the mass M (the Earth). We are interested in the 1-component of E at a point with co-ordinates $(0, r-d, 0)$—i.e. at a point *a* co-ordinate distance d from the center of the wire in the negative direction of the x^2-axis.

Now in general we have

$$E = -\nabla A^4 - \frac{1}{C}\frac{\partial A}{\partial t} \tag{3a.1}$$

Therefore, the component of E 'parallel' to the wire is given by

$$E_1 = -A^4_{,1} = -A_{4,1} \tag{3a.2}$$

since the situation is static. The word parallel here is in quotes since, because of the curved nature of the space, the two x^1-co-ordinate lines involved are not actually parallel. However, by making d sufficiently small (and/or r_0 sufficiently large) we can speak of E_1 as the component of E truly parallel to the wire to any degree of precision that we desire.

We see, then, that to find E_1 we must find A_4.

Referring then to equation (3.22) we see that the terms involving ∇A_4 and $j_4(GM/rC^2)$ are small perturbations to an equation whose solution we know. That is, without these terms being present, the solution to equation (3.22) is

$$A_4^{(0)} = \frac{1}{C}\int \frac{j_4' \, d^3 x'}{r'} \tag{3a.3}$$

where the integration is along the wire, and r' is the co-ordinate distance from the general point on the wire to the field point.

We now assume that the solution, A_4, to the wire equation with the perturbation terms present is of the form

$$A_4 = A_1^{(0)} + \alpha A_4^{(1)} + \cdots \tag{3a.4}$$

where

$$\alpha \equiv \frac{2GM}{r_0 C^2} \tag{3a.5}$$

Inserting this series into equation (3.22) we obtain the following equation, which is correct to first order in α:

$$\nabla^2 A_4 + \frac{\alpha}{2} \frac{r_0}{r^3 C} \mathbf{r} . \nabla \int \frac{j_4' \, d^3 x'}{r'} = -\frac{4\pi}{C}\left(1 + \alpha \frac{r_0}{r}\right) j_4 \tag{3a.6}$$

where primes have been appended to all quantities involved in an integration, to avoid confusion.

From the above relation we may immediately write for A_4,

$$A_4 = \frac{1}{4\pi} \int \left\{ j_4' \frac{4\pi}{C}\left(1 + \frac{\alpha r_0}{r}\right) + \tfrac{1}{2}\alpha \frac{r_0}{r^3 C} \mathbf{r} . \nabla \int \frac{j_4'' \, d^3 x''}{r''} \right\} \frac{d^3 x'}{r'} \tag{3a.7}$$

where all integrations are over the wire; r' and r'' signify distances from points on the wire to the field point; r denotes the co-ordinate distance from the center of M to points on the wire; and $j_4' = j_4'' = \text{const}$.

It is not difficult (but rather lengthy) to evaluate the above expression, especially if the radius of the wire is sufficiently small. For our purpose, however, we need only note that the above expression must be an even function of x^1. From this it follows that

$$E_1\Big|_{x^1=0} = -A^4_{,1}\Big|_{x^1=0} = 0 \tag{3a.8}$$

and this is to first order in $\alpha = 2GM/r_0 C^2$.

Therefore, the generally covariant Maxwell's equations lead to the result that the component of E parallel to a 'straight' resistanceless current-carrying wire segment in a weak, static, spherically symmetric gravitational field is zero to terms of first order in $GM/r_0 C^2$ (at a point equidistant from the ends of the wire)—thus contradicting the Principle of Equivalence.

It is to be noted here that this result persists for vanishingly small gravitational fields where space becomes Galilean, since the difference between the two predicted fields divided by either does *not* go to zero in the limit of large r_0.

Furthermore, we note that the zero value of the parallel component of E (through first order in α) predicted to exist in the Schwarszchild co-ordinate system for large r_0, implies that this component is also zero to first order in α in a local inertial system located at very large r_0. This is so because, for large r_0, the ratio of E in the Schwarszchild co-ordinate system to E in a local inertial system at the same location is expected to differ from unity by a term the order of α. This rules out the possibility that the predicted zero component was merely the result of the chosen co-ordinate system.

4. *Discussion*

It has been demonstrated that the covariant formulation of electrodynamics found in General Relativity and the Principle of Equivalence as formulated here are incompatible. The nature of the disagreement is one that, *in principle*, could be checked by experiment as follows: One would take a straight superconducting wire segment (about 1 m long) placed horizontally in the Earth's field. Since the wire is a superconductor we would expect classically that E outside be absolutely zero (assuming that the E field due to the rest of the circuit was screened off of course). Now, on the one hand, the Principle of Equivalence predicts an electric field component, in the vicinity of the wire, parallel to the wire in this case—which could be displayed as a potential difference on a nearby parallel wire segment or as dipole radiation from this segment if the current in the superconductor is alternating. On the other hand, General Relativity seems to imply that E_1 be zero in this case (at a point equidistant from the wire ends). A delicate measurement here could resolve the issue.

Appendix

Consider a wire of length l carrying a constant current i and moving with constant intrinsic acceleration g. Let this wire be viewed from an inertial system relative to which the wire is at rest at time $t = 0$. All quantities shall refer to measurements made in this inertial system. The motion of the wire relative to this Lorentz frame is as follows: The wire approaches the frame with decreasing speed and finally stops and then reverses its direction of motion. We wish to find E_1 at point P which is fixed in the inertial frame and which is equidistant from the ends of the wire and is a distance ζ_0 (denoted by d in the text) from the wire at time $t = 0$. We take the conventional current in the wire to be in the direction of the $+x^1$-axis, and the acceleration to be along the $+x^3$-axis—see Fig. 1.

Now,

$$E_1 = -\frac{1}{C}\dot{A}_1 \qquad (\text{since } \phi_{,1=0}) \tag{A.1}$$

and

$$A_1 = \frac{i}{C}\int_{\Gamma} \frac{dx_1}{r\left(1 - \dfrac{v\cos\theta}{rC}\right)} \tag{A.2}$$

where Γ denotes the retarded contour of the wire, and all quantities under the integral sign are to be retarded.

Therefore,

$$E_1 = -\frac{i}{C^2}\left\{\frac{d}{dt}\int_{\Gamma} \frac{dx_1}{r\left(1 - \dfrac{v\cos\theta}{rC}\right)}\right\}_{t=0} \tag{A.3}$$

Since E_1 depends on a time derivative we shall first be interested in evaluating A_1 for very small positive t, at point P, and then taking the derivative at $t=0$. For this reason, in the following, we only retain terms involving small positive t, to first order.

Now,

$$v_3(t) = \frac{Cgt}{\sqrt{(C^2+g^2t^2)}}, \qquad v_3(0)=0, \qquad v_3(<0)<0 \tag{A.4}$$

Assuming that $gl \ll C^2$, it follows that $gt \ll C$, and therefore that

$$v_3(t) \cong gt \tag{A.4'}$$

for $t<0$.

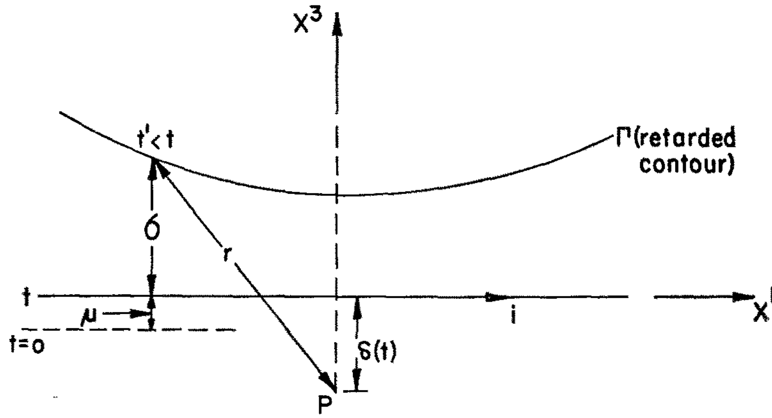

Figure 1.—Contribution to field at point P at time t.

Therefore,

$$-\int_{t'}^{0} v_3(\xi)\,d\xi + \int_{0}^{t} v_3(\xi)\,d\xi = \sigma + \mu = \frac{g}{2}\{t^2+(t')^2\} \tag{A.5}$$

and

$$\frac{r}{C} = (t-t') \tag{A.6}$$

where t is small and positive; σ and $\sigma+\mu$ are distances to the location of the wire at times t and $t=0$ respectively; t' is the retarded time, for a segment at a given x_1 along the wire, for the field at P at time t.

Now,

$$r^2 - x_1{}^2 = [\sigma+\zeta(t)]^2 \tag{A.7}$$

where $\zeta(t)$ is the distance of the wire from P at time t.

We also have the relation

$$\zeta(t) \cong \zeta_0 + \tfrac{1}{2}gt^2 \cong \zeta_0 \tag{A.8}$$

Now from (A.5) we have

$$t' \cong -\sqrt{\left(\frac{2\sigma}{g}\right)} \tag{A.9}$$

since $\mu \cong \frac{1}{2}gt^2 \cong 0$.

From (A.6) we now have

$$\frac{r}{C} \cong t+\sqrt{\left(\frac{2\sigma}{g}\right)} \tag{A.10}$$

Using this relation in (A.7) then yields the relation

$$r^2 - x_1{}^2 \cong \zeta_0{}^2 + g\zeta_0\left\{\frac{r^2}{C^2} - \frac{2rt}{C}\right\} + \frac{g^2}{4}\left\{\frac{r^4}{C^4} - \frac{4r^3 t}{C^3}\right\} \tag{A.11}$$

Now we require that $\zeta \ll l$. Since we also have $gl \ll C^2$, this implies that

$$\frac{gl}{C^2}\frac{l}{\zeta_0} \ll \frac{l}{\zeta_0} \tag{A.12}$$

For fixed l and ζ_0 we require that g be sufficiently small that we can still have

$$\frac{gl}{C^2}\frac{l}{\zeta_0} \ll 1 \tag{A.13}$$

Now then,

$$\frac{g^2}{\zeta_0}\frac{r^4/C^4}{gr^2/C^2} \cong \frac{gl}{C^2}\frac{l}{\zeta_0} \ll 1; \qquad \frac{g^2}{\zeta_0}\frac{r^3 t/C^3}{grt/C} \cong \frac{gl}{C^2}\frac{l}{\zeta_0} \ll 1 \tag{A.14}$$

Therefore, (A.11) can be written as

$$r^2\left(1 - \frac{\zeta_0 g}{C^2}\right) \cong x_1{}^2 + \zeta_0{}^2 - \frac{2\zeta_0 grt}{C} \tag{A.15}$$

Now,

$$\frac{\zeta_0 g}{C^2} \ll 1 \qquad (\text{since } \zeta_0 \ll l) \tag{A.16}$$

Therefore, we have

$$r^2 \cong x_1{}^2 + \zeta_0{}^2 - \frac{2\zeta_0 grt}{C} \tag{A.17}$$

which yields the following solution for r

$$r \cong -\frac{\zeta_0 gt}{C} + \sqrt{(x_1{}^2 + \zeta_0{}^2)} \tag{A.18}$$

Now since $v/C \ll 1$, we can write

$$E_1 \cong -\frac{i}{C^2}\left\{\frac{d}{dt}\int_\Gamma \frac{dx_1}{r}\right\}_{t=0} = -\frac{i}{C^2}\left\{\frac{d}{dt}\int_{-l/2}^{l/2} \frac{dx_1}{-\frac{\zeta_0 gt}{C} + \sqrt{(x_1{}^2 + \zeta_0{}^2)}}\right\}_{t=0} \tag{A.19}$$

Therefore, we finally have the relation

$$E_1 \cong -\frac{2ig}{C^3}\tan^{-1}\frac{l}{2d} \cong -\frac{\pi ig}{C^3} \tag{A.20}$$

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